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Pre-apartness structures on spaces of functions

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Abstract

Pre-apartness structures are defined on Y^X, where X is an inhabited set and Y a uniform space. These structures clarify the discussion of proximal and uniform convergence in the constructive theory of apartness spaces.

Keywords: Constructive; Apartness; Proximal convergence; Uniform structure

1. Introduction

Although the convergence of nets of functions into proximity spaces has been investigated by several authors [12,15,14], we have been unable to find any discussion of proximity structures defined on function spaces. In this paper, we consider structures on Y^X within the theory of apartness spaces, a constructive counterpart of the classical theory of proximity spaces. We show that when X is an inhabited set (that is, one for which we can construct elements) and Y is a uniform space, the function space Y^X can be equipped with pre-apartness structures that provide natural frameworks for the discussion of proximal and uniform convergence.

Perhaps the most distinctive feature of our work is that it is fully constructive—in other words, we use only intuitionistic logic. By so doing, we ensure that all our proofs embody algorithms; moreover, the proofs themselves verify that the algorithms meet their specifications. We do not make any requirements, such as those used in intuitionistic mathematics or recursive function theory, beyond that of the exclusive use of intuitionistic logic. For this reason, all our work is valid in intuitionistic mathematics, under a recursive interpretation, and in classical

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(that is, standard) mathematics. Moreover, as Andrej Bauer informed the CCA 05 meeting in Kyoto [2], our work can readily be translated into the framework of Weihrauch's Type II Effectivity Theory, one of the most important approaches to computable analysis. We refer the reader to [3,5,21,24] for more information about constructive/computable mathematics.

On the other hand, the theory of apartness has potential for applications in computer science. We are currently investigating its use in the region connection calculus (see [19,20]), as well as a version of the theory of apartness in Boolean lattices [8], in the spirit of Sambin and Martin-Löf's formal topology [16].

2. Basic notions

Let X be a set with an inequality relation $\neq$ satisfying the properties

$$x \neq y \implies \neg(x = y),$$
$$x \neq y \implies y \neq x.$$

Note that the inequality $\neq$ need not be the denial inequality, in which

$$x \neq y \iff \neg(x = y).$$

For example, the standard inequality on a metric space (X, ρ) is defined by

$$x \neq y \iff \rho(x, y) > 0$$

and is equivalent to the denial inequality if and only if we accept Markov's principle,

> For each binary sequence $(a_n)_{n \geqslant 1}$, if it is impossible that $a_n = 0$ for all n, then there exists n such that $a_n = 1$.

Since this principle represents a form of unbounded search and is independent of Heyting arithmetic (Peano arithmetic with intuitionistic logic), it is normally regarded as essentially nonconstructive.

Returning to a general set X with an inequality, we assume, to avoid pseudo-generality, that X is inhabited. In the theory of apartness spaces, the inequality on X is generalised to either a relation between points and sets or else a relation between sets and sets, as follows.

A *point-set pre-apartness* is a relation $\bowtie$ between points $x \in X$ and sets $S \subset X$ that satisfies the following four axioms: [1]

A1 $x \bowtie \emptyset$.
A2 $-S \subset {\sim}S$.
A3 $x \bowtie (S \cup T) \iff x \bowtie S \wedge x \bowtie T$.
A4 $x \bowtie S \wedge -S \subset {\sim}T \implies x \bowtie T$.

In these axioms and later, ${\sim}S$ stands for the *complement*,

$$\left\{x \in X : \forall_{y \in S}\,(x \neq y)\right\},$$

of a subset S of X, and $-S$ for the *apartness complement*,

$$\{x \in X : x \bowtie S\}$$

[1] For more detailed information about apartness see [6,7,9,18].

of S. These two types of complement should not be confused with the *logical complement*

$$\neg S = \{x \in X : x \notin S\}$$

of S. Note that under axioms **A1**–**A4** we have $-S \subset {\sim}S \subset \neg S$.

The set X together with a point-set pre-apartness is called a *point-set pre-apartness space*. If, in addition, the relation $\bowtie$ satisfies the axiom

A5 $x \bowtie S \Longrightarrow \forall_{y \in X}\, (x \neq y \vee y \bowtie S)$,

then we call $\bowtie$ a *point-set apartness* on X, and X a *point-set apartness space*.

The unions of apartness complements in a point-set pre-apartness space X form a topology, the *apartness topology* $\tau_{\bowtie}$.

By a *directed set* we mean an inhabited set D with a preorder[2] $\succcurlyeq$ such that for all $m, n \in D$ there exists $p \in D$ with $p \succcurlyeq m$ and $p \succcurlyeq n$. A *net* in a set X is a mapping $n \rightsquigarrow x_n$ of such a set D into X, and is normally denoted by $(x_n)_{n \in D}$. A net $(x_n)_{n \in D}$ in a point-set pre-apartness space X is said to *converge*, or to be *apartness convergent*, to the *limit* $x \in X$ if

$$\forall_{S \subset X} \left(x \bowtie S \Longrightarrow \exists_{N \in D} \forall_{n \succcurlyeq N}\, (x_n \bowtie S) \right).$$

This notion is equivalent to that of convergence in the apartness topology $\tau_{\bowtie}$.

For our purposes, a *set–set pre-apartness* on X is a symmetric[3] relation $\bowtie$ between subsets of X that satisfies these axioms:

B1 $X \bowtie \emptyset$.
B2 $S \bowtie T \Longrightarrow S \subset {\sim}T$.
B3 $R \bowtie (S \cup T) \Longleftrightarrow R \bowtie S \wedge R \bowtie T$.
B4 $-S \subset {\sim}T \Longrightarrow -S \subset -T$,
where $-S$ is the apartness complement derived from with the *associated point-set pre-apartness* defined by

$$x \bowtie S \Longleftrightarrow \{x\} \bowtie S. \tag{1}$$

Equipped with a set–set pre-apartness, the set X is called a *set–set pre-apartness space*. If, in addition, $\bowtie$ satisfies

B5 $x \bowtie S \Longrightarrow \exists_{T \subset X} (x \bowtie T \wedge \forall_{y \in X} (y \bowtie S \vee y \in T))$,
we say that it is a *set–set apartness* and that X is a *set–set apartness space*. In that case, the relation defined at (1) is a point-set apartness.

When we use point-set names and notations in the context of a set–set pre-apartness, we are implicitly referring to the associated point-set pre-apartness.

Finally, if the relation $\bowtie$ satisfies axioms **B1**, **B3**, **B4**, and

B2$_w$ $S \bowtie T \Longrightarrow S \cap T = \emptyset$,

then we call it a *weak pre-apartness* and we call X a *weak pre-apartness space*.

[2] The classical theory of nets requires a partial order. If we used a partial order in our constructive theory, we would run into difficulties that the classical theory avoids by applications of the axiom of choice (which is essentially nonconstructive—see [10,11]).

[3] In our monograph [9] we drop the requirement of symmetry.

3. A pre-apartness for proximal convergence

Let X be an inhabited set, and $(Y, \bowtie)$ a set–set pre-apartness space. The inequality on Y^X is defined by

$$f \neq g \Longleftrightarrow \exists_{x \in X} (f(x) \neq g(x)) .$$

For all $A \subset X$ and $B \subset Y$ define

$$U_{A,B} = \left\{ f \in Y^X : f(A) \bowtie B \right\} .$$

The sets $U_{A,B}$, with $A \subset X$ and $B \subset Y$, form a subbase of a topology τ_p on Y^X called the *topology of proximal convergence*. (The reason for this name will become apparent once we reach Proposition 3.) Following the standard construction of a pre-apartness from a topology [7], we obtain from this topology a point-set pre-apartness on Y^X by setting $f \bowtie_{Y^X} S$ if and only if there exist finitely many subsets $A_1, \ldots, A_m$ of X, and finitely many subsets $B_1, \ldots, B_m$ of Y, such that

$$f \in \bigcap_{i=1}^{m} U_{A_i, B_i} \subset {\sim} S.$$

It is easy to show that $\bowtie_{Y^X}$ is a pre-apartness on Y^X. Moreover, if $f, g \in Y^X$ and $f \bowtie_{Y^X} \{g\}$, then $f \in {\sim}\{g\}$, so $f \neq g$.

The following Brouwerian example shows that in general we cannot prove that $\bowtie_{Y^X}$ is a full-blooded apartness on Y^X. More precisely, it shows constructively that we cannot prove **A5** for $\bowtie_{Y^X}$ even when Y is a two-point apartness space. Let $X = \mathbb{N}$ and $Y = \{0, 1\}$, and let $f \in Y^X$ be the constant function 1. Given an increasing binary sequence $(a_n)_{n \geqslant 0}$, define

$$g(n) = \min \{ f(n), 1 - a_n \} .$$

Define also

$$S = \left\{ h \in Y^X : \exists_N \forall_{n \geqslant N} (h(n) = 0) \right\} .$$

Then $f \in U_{X,\{0\}} \subset {\sim} S$, so $f \bowtie_{Y^X} S$. If $f \neq g$, then there exists n with $a_n = 1$. On the other hand, if $a_n = 1 - a_{n-1}$, then $g \in S$; so if $g \bowtie_{Y^X} S$, then $a_n = 0$ for all n. Thus if $\{0, 1\}^{\mathbb{N}}$ satisfies **A5**, then we can prove the following essentially nonconstructive principle:

LPO $\forall_{\mathbf{x} \in 2^{\mathbb{N}}} (\forall_n (x_n = 0) \vee \exists_n (x_n = 1))$.

For the record, we prove a couple of simple results about separation properties of the pre-apartness on a function space. For these we define a point-set pre-apartness space $(X, \bowtie)$ to be

- **T**$_1$ if

$$\forall_{x \in X} \forall_{y \in X} (x \neq y \Longrightarrow x \bowtie \{y\}) ;$$

- **Hausdorff** if for all $x, y \in X$ with $x \neq y$, there exist $U, V \subset X$ such that $x \in -U$, $y \in -V$, and $-U \subset {\sim} - V$.

Thus X is T_1 (resp., Hausdorff) if and only if its apartness topology is T_1 (resp., Hausdorff).

Proposition 1. *Let X be an inhabited set, and Y a T_1 pre-apartness space. Then $\left(Y^X, \tau_p\right)$ is T_1.*

Proof. If $f, g \in Y^X$ and $f \neq g$, then there exists $x \in X$ such that $f(x) \neq g(x)$; so, since the apartness space Y is T_1, $f(x) \bowtie \{g(x)\}$ and therefore $f \in U_{\{x\},\{g(x)\}}$. But if $h \in U_{\{x\},\{g(x)\}}$, then $h(x) \bowtie \{g(x)\}$ and therefore, by **A2** in Y, $h(x) \neq g(x)$; whence $U_{\{x\},\{g(x)\}} \subset {\sim}\{g\}$. Thus $f \bowtie_{Y^X} g$. $\square$

Proposition 2. *Let X be an inhabited set, and Y a Hausdorff pre-apartness space. Then $\left(Y^X, \tau_p\right)$ is Hausdorff.*

Proof. If $f \neq g$ in Y^X, then we can find $x \in X$ such that $f(x) \neq g(x)$. Since Y is Hausdorff, there exist subsets S, T of Y such that $f(x) \in -S$, $g(x) \in -T$, and $-S \subset \sim -T$. Then

$$f \in U_{\{x\},S} \subset {\sim}{\sim}U_{\{x\},S}$$

and therefore $f \in -{\sim}U_{\{x\},S}$. Similarly, $g \in -{\sim}U_{\{x\},T}$. Moreover, for each $h \in U_{\{x\},S}$ we have $h(x) \in -S \subset \sim -T$; whence $h \in {\sim}U_{\{x\},T}$. Thus $U_{\{x\},S} \subset {\sim}U_{\{x\},T}$. It follows that $-{\sim}U_{\{x\},T} \subset -U_{\{x\},S} \subset {\sim}U_{\{x\},S}$ and hence that

$$-{\sim}U_{\{x\},S} \subset -\left(-{\sim}U_{\{x\},T}\right) \subset \sim\left(-{\sim}U_{\{x\},T}\right). \qquad \square$$

A net $(f_n)_{n\in D}$ of elements of the function space Y^X is said to be *proximally convergent* to an element f of Y^X if[4]

$$\forall_{A\subset X}\forall_{B\subset Y}\left(f(A)\bowtie B \Longrightarrow \exists_{N\in D}\forall_{n\succcurlyeq N}\left(f_n(A)\bowtie B\right)\right).$$

We now explore the connection between proximal convergence on the one hand, and convergence relative to the topology τ_p and the pre-apartness $\bowtie_{Y^X}$ on the other.

Proposition 3. *Let X be an inhabited set, and Y a pre-apartness space. Let $(f_n)_{n\in D}$ be a net in Y^X, and $f \in Y^X$. Then (f_n) converges to f proximally if and only if it converges to f in the topology τ_p.*

Proof. If $(f_n)_{n\in D}$ converges proximally to f in Y^X, and U is a τ_p-open subset of Y^X containing f, then there exist subsets $A_1, \ldots, A_m$ of X, and subsets $B_1, \ldots, B_m$ of Y, such that $f \in \bigcap_{i=1}^{m} U_{A_i,B_i} \subset U$. For each i, since $f(A_i)\bowtie B_i$, there exists $n_i \in D$ such that $f_n \in U_{A_i,B_i}$ whenever $n \succcurlyeq n_i$. Choosing $N \in D$ such that $N \succcurlyeq n_i$ for each i $(1 \leqslant i \leqslant m)$, we see that if $n \succcurlyeq N$, then $f_n \in \bigcap_{i=1}^{m} U_{A_i,B_i} \subset U$. Thus $(f_n)_{n\in D}$ is τ_p-convergent to f.

Conversely, if $(f_n)_{n\in D}$ converges to f relative to the topology τ_p, and if $f(A)\bowtie B$, then f belongs to the τ_p-open set $U_{A,B}$; so $f_n \in U_{A,B}$ —that is, $f_n(A)\bowtie B$ —eventually. $\square$

How does proximal convergence relate to convergence relative to the pre-apartness $\bowtie_{Y^X}$? To answer this, we first introduce some definitions and prove a general result about convergence.

[4] For more on proximal convergence see [22,23].

A *topological pre-apartness space* is a topological space (X, τ) taken with the point-set relation (easily seen to be a point-set pre-apartness) $\bowtie$ defined by

$$x \bowtie S \iff \exists_{U \in \tau} (x \in U \subset \sim S).$$

For example, the topological space $\left(Y^X, \tau_p\right)$ gives rise to the topological pre-apartness space $\left(Y^X, \bowtie_{Y^X}\right)$ in this way.

A topological pre-apartness space is said to be *topologically consistent* if the apartness topology $\tau_{\bowtie}$ corresponding to $\bowtie$ coincides with the original topology τ. Although we always have $\tau \subset \tau_{\bowtie}$, and the reverse inclusion holds classically, we cannot be certain constructively that the two topologies coincide; see [7,9]. Our next proposition gives a necessary and sufficient condition for a topological pre-apartness space to be topologically consistent, and requires the following definitions.

For each x in a pre-apartness space $(X, \bowtie)$ let

$$D_x = \{(\xi, U) : x \in -U \wedge \xi \in -U\},$$

with equality defined by

$$(\xi, U) = \left(\xi', U'\right) \iff \left(\xi = \xi' \wedge -U = -U'\right),$$

and for each $n = (\xi, U)$ in D_x define $x_n = \xi$. It is easy to see that D_x is a directed set under the *reverse inclusion preorder* defined by

$$(\xi, U) \succcurlyeq \left(\xi', U'\right) \iff -U \subset -U',$$

so that $\mathcal{N}_x = (x_n)_{n \in D_x}$ is a net—the *basic neighbourhood net*[5] of x.

It is trivial to prove that in a topological pre-apartness space (X, τ),

- topological convergence implies $\bowtie_\tau$-convergence;
- if (X, τ) is topologically consistent, then $\bowtie_\tau$-convergence implies topological convergence.

Conversely, we have:

Proposition 4. *Let (X, τ) be a topological pre-apartness space such that $\bowtie_\tau$-convergence implies topological convergence. Then X is topologically consistent.*

Proof. Consider a $\tau_{\bowtie}$-open set $U \subset X$ and any $x \in U$. Let $(x_n)_{n \in D_x}$ be the basic neighbourhood net of x, which converges to x relative to the pre-apartness. By our supposition, this net is topologically convergent to x; so there exists $N = (\xi, W) \in D_x$ such that $x_n \in U$ for all $n \succcurlyeq N$. For each $y \in -W$ we have $(y, W) \succcurlyeq (\xi, W)$, by definition of the reverse inclusion preorder; so $y = x_{(y,W)} \in U$. Hence $-W \subset U$. It follows that U is a union of apartness complements, and is therefore nearly open. $\square$

Corollary 1. *Let X be an inhabited set, and Y a pre-apartness space. Then proximal convergence in Y^X implies $\bowtie_{Y^X}$-convergence; but the converse holds if and only if $\left(Y^X, \tau_p\right)$ is topologically consistent.*

[5] Our definition of the basic neighbourhood net of x avoids any use of the Axiom of Choice [10,11].

Proof. This follows from Propositions 3 and 4, and the remarks preceding the latter. □

To end this section, we show that the pre-apartness structure on Y^X has a natural categorical property.

Proposition 5. *Let X be an inhabited set, and Y a pre-apartness space. For each $x \in X$ the* evaluation map $\hat{x} : f \rightsquigarrow f(x)$ *is continuous in the sense that*

$$\forall_{f \in Y^X} \forall_{S \subset Y^X} \left(\hat{x}(f) \bowtie_Y \hat{x}(S) \Longrightarrow f \bowtie_{Y^X} S \right).$$

Proof. Suppose that $\hat{x}(f) \bowtie_Y \hat{x}(S)$, where $f \in Y^X$ and $S \subset Y^X$. Let

$$A = \{x\}, \quad B = \hat{x}(S).$$

Then, by our supposition, $f \in U_{A,B}$. On the other hand, if $g \in U_{A,B}$, then $g(x) \bowtie \{h(x) : h \in S\}$; whence $g \in \sim S$. Thus $f \in U_{A,B} \subset \sim S$ and therefore $f \bowtie_{Y^X} S$. □

4. A weak pre-apartness for uniform convergence

In this section we consider Y^X when X is an inhabited set and Y carries a uniform structure.

Let E be an inhabited set, and U, V subsets of the Cartesian product $E \times E$. We define certain associated subsets as follows:

$$U \circ V = \{(x, y) : \exists z \in E((x, z) \in U \wedge (z, y) \in V)\},$$
$$U^1 = U, \quad U^{n+1} = U \circ U^n \quad (n = 1, 2, \ldots),$$
$$U^{-1} = \{(x, y) : (y, x) \in U\},$$
$$U[x] = \{y \in E : (x, y) \in U\}.$$

We call U *symmetric* if $U = U^{-1}$.

We say that a family $\mathcal{U}$ of subsets of E is a *pre-uniform structure*, or a *pre-uniformity*, on E if the following four conditions hold:

- Every finite intersection of sets in $\mathcal{U}$ belongs to $\mathcal{U}$.
- Every subset of $E \times E$ that contains a member of $\mathcal{U}$ is in $\mathcal{U}$.
- Every $U \in \mathcal{U}$ includes the *diagonal* $\Delta = \{(x, x) : x \in E\}$ of $X \times X$.
- For each $U \in \mathcal{U}$, $U^{-1} \in \mathcal{U}$ and there exists $V \in \mathcal{U}$ such that $V^2 \subset U$.

The elements of $\mathcal{U}$ are called the *entourages* of (the pre-uniform structure on) E, and the pair $(E, \mathcal{U})$—or simply E itself—is called a *pre-uniform space*. We define the standard inequality relation on E by

$$\forall_{x,y \in E} \left(x \neq y \Longleftrightarrow \exists_{U \in \mathcal{U}} \left((x, y) \notin U\right)\right),$$

and a binary relation $\bowtie_{\mathcal{U}}$ between subsets of E by

$$S \bowtie_{\mathcal{U}} T \Longleftrightarrow \exists_{U \in \mathcal{U}} (S \times T \subset \neg U).$$

Then $\bowtie_{\mathcal{U}}$ is a pre-apartness on E. The only property of a pre-apartness whose verification is a little delicate in this case is **B4**, so we present the detail. Let S, T be subsets of E such that $-S \subset \sim T$, and consider any $x \in -S$. Choose $U \in \mathcal{U}$ such that $\{x\} \times S \subset \neg U$, and then $V \in \mathcal{U}$ such that

$V^2 \subset U$. Given $t \in T$, suppose that $(x,t) \in V$. If $s \in S$ and $(t,s) \in V$, then $(x,s) \in V^2 \subset U$, a contradiction of our choice of U. Hence $(t,s) \notin V$ for all $s \in S$, and therefore $\{t\} \times S \subset \neg V$. Thus $t \in -S \subset \sim T$, which is again a contradiction. We conclude that $(x,t) \notin V$. Since $t \in T$ was arbitrary, it follows that $\{x\} \times T \subset \neg V$ and hence that $x \in -T$.

The subsets of the pre-uniform space E that have the form

$$U[y] = \left\{ y' \in E : \left(y, y'\right) \in U \right\}$$

with $U \in \mathcal{U}$ form a base of neighbourhoods of y in the *uniform topology* on E.

We call the pre-uniform structure $\mathcal{U}$ a *uniform structure* if it satisfies [6]

(**U**) For each $U \in \mathcal{U}$ there exists $V \in \mathcal{U}$ such that $E \times E = U \cup \neg V$.

We then say that $(E, \mathcal{U})$, or just E itself, is a *uniform space*. For example, a metric space (E, ρ) is a uniform space in which the uniformity consists of all subsets of $E \times E$ that contain sets of the form $\{(x,y) : \rho(x,y) \leqslant \varepsilon\}$ for some $\varepsilon > 0$.

We note two consequences of property (**U**) in a uniform space E:

- $\forall_{x,y \in E}\, (x \neq y \Longleftrightarrow \exists_{U \in \mathcal{U}}\, ((x,y) \in \sim U))$.
- For each $U \in \mathcal{U}$ and each positive integer n there exists an n-**chain** of entourages of E: that is, an n-tuple $(U_1, \ldots, U_n)$ such that $U_1 = U$ and for each $k \geqslant 2$,

$$U_k \in \mathcal{U},\ U_k = U_k^{-1},\ U_k \circ U_k \subset U_{k-1} \quad \text{and} \quad E \times E = U_{k-1} \cup \sim U_k.$$

Now let $(Y, \mathcal{U})$ be a uniform space, and for each $U \in \mathcal{U}$ define

$$W_X(U) = \left\{ (f,g) \in Y^X \times Y^X : \forall_{x \in X}\, ((f(x), g(x)) \in U) \right\}.$$

The set

$$\mathcal{W} = \{W_X(U) : U \in \mathcal{U}\}$$

satisfies the conditions for a pre-uniform structure. Classically, it also satisfies (**U**); however, as we shall see below, if $\left(Y^X, \mathcal{W}\right)$ satisfies (**U**), then LPO holds, so for us $\mathcal{W}$ is just a pre-uniform structure on Y^X.

The pre-uniform structure $\mathcal{W}$ gives rise to a topology $\tau_{\mathcal{W}}$ on Y^X, in which the sets

$$W_X(U)[f] = \left\{ g \in Y^X : (f,g) \in W_X(U) \right\}$$

form a base of neighbourhoods of f. A net $(f_n)_{n \in D}$ converges to f in this topology if and only if for each $U \in \mathcal{U}$ there exists $N \in D$ such that $(f, f_n) \in W_X(U)$—that is,

$$\forall_{x \in X}\, ((f(x), f_n(x)) \in U)$$

—for all $n \succcurlyeq N$. In other words, convergence with respect to $\tau_{\mathcal{W}}$ is just what we already know as uniform convergence. For this reason we call $\tau_{\mathcal{W}}$ the *topology of uniform convergence* on Y^X.

Corresponding to the pre-uniform structure $\mathcal{W}$ is a relation $\bowtie_{\mathcal{W}}$ on subsets of Y^X, defined by

$$S \bowtie_{\mathcal{W}} T \Longleftrightarrow \exists_{U \in \mathcal{U}}\, (S \times T \subset \neg W_X(U)).$$

[6] Classically, property (**U**) always holds with $V = U$. It appears to be important to postulate it in the constructive theory.

We show that $\bowtie_{\mathcal{W}}$ is a weak pre-apartness. First, since

$$Y^X \times \emptyset = \emptyset \subset \neg\left(Y^X \times Y^X\right)$$

and $Y^X \times Y^X = W_X(Y \times Y) \in \mathcal{W}$, axiom **B1** holds. To handle **B2**$_w$, let $S \bowtie_{\mathcal{W}} T$ and choose $U \in \mathcal{U}$ such that $S \times T \subset \neg W_X(U)$; if $f \in S \cap T$, then $(f, f) \in S \times T$ and thus $(f, f) \in \neg W_X(U)$, which is absurd since $W_X(U)$ includes the diagonal of $Y^X \times Y^X$.

Next let $R \bowtie (S \cup T)$ and choose $U \in \mathcal{U}$ such that $R \times (S \cup T) \subset \neg W_X(U)$. Then, trivially, $R \times S \subset \neg W_X(U)$ and $R \times T \subset \neg W_X(U)$. Thus $R \bowtie S$ and $R \bowtie T$. Suppose, conversely, that $R \bowtie S$ and $R \bowtie T$. Then there exist $U, V \in \mathcal{U}$ such that $R \times S \subset \neg W_X(U)$ and $R \times T \subset \neg W_X(V)$; whence

$$\begin{aligned} R \times (S \cup T) &\subset \neg W_X(U) \cup \neg W_X(V) \\ &\subset \neg\left(W_X(U) \cap W_X(V)\right) = \neg W_X(U \cap V) \end{aligned}$$

and therefore $R \bowtie (S \cup T)$. This completes the verification of **B3**. To deal with **B4/A4**, we actually prove a stronger result: namely,

$$-S \subset \neg T \Longrightarrow -S \subset -T.$$

First note that

$$f \bowtie_{\mathcal{W}} S \Longleftrightarrow \exists_{U \in \mathcal{U}} \left(W_X(U)[f] \subset \neg S\right).$$

Let $f \bowtie_{\mathcal{W}} S$ and $-S \subset \neg T$. Choose $U \in \mathcal{U}$ such that $W_X(U)[f] \subset \neg S$, and then $V \in \mathcal{U}$ such that $V \circ V \subset U$. Consider $g \in W_X(V)[f]$ and $h \in S$. If $(g, h) \in W_X(V)$, then

$$(f, h) \in W_X(V) \circ W_X(V) \subset W_X(U),$$

so $h \in W_X(U)[f] \subset \neg S$, a contradiction. Hence $(g, h) \notin W_X(V)$. It follows that $\{g\} \times S \subset \neg W_X(V)$ and therefore that $g \in -S$. Since $g \in W_X(V)[f]$ is arbitrary, we conclude that $W_X(V)[f] \subset -S$. Hence

$$f \in W_X(V)[f] \subset -S \subset \neg T$$

and therefore $f \bowtie_{\mathcal{W}} T$. This completes the verification of **B4**.

What about **B5**? The Brouwerian example that we considered in Section 2 also shows that we cannot prove even this weak version of **A5**,

$$x \bowtie S \Longrightarrow \forall_{y \in X} \left(\neg(x = y) \vee y \bowtie S\right), \tag{2}$$

a consequence of **B5**, for the relation $\bowtie_{\mathcal{W}}$; whence a fortiori we cannot prove **B5** itself. Indeed, taking $X = \mathbb{N}$ and $Y = \{0, 1\}$, we see that the uniform structure on the discrete space Y consists of all supersets of the diagonal

$$\Delta = \{(0, 0), (1, 1)\}.$$

Thus in this case, $\mathcal{W}$ consists of all sets $W_X(U)$ where $\Delta \subset U$. With f, $(a_n)_{n \geqslant 0}$, g and S as in the Brouwerian example, for each $\phi \in S$ there exists n such that $\phi(n) = 0 \neq f(n)$; whence $(f, \phi) \in \sim W_X(\Delta) \subset \neg W_X(\Delta)$ and therefore $f \bowtie_{\mathcal{W}} S$. As before, if $f \neq g$, then there exists n with $a_n = 1$; whereas if $g \bowtie_{\mathcal{W}} S$, then $g \notin S$, so $a_n = 0$ for all n. Thus if $\left(\{0, 1\}^{\mathbb{N}}, \bowtie_{\mathcal{W}}\right)$

satisfies (2), then LPO holds. It follows a fortiori that if $\left(\{0,1\}^{\mathbb{N}}, \mathcal{W}\right)$ is a uniform space—that is, satisfies (**U**)—then LPO holds.

What, if any, are the connections between τ_p and $\tau_{\mathcal{W}}$, between $\bowtie_{Y^X}$ and $\bowtie_{\mathcal{W}}$, and between various notions of convergence associated with those structures on Y^X?

Proposition 6. *Let X be an inhabited set, and $(Y, \mathcal{U})$ a uniform space. Then the topology of uniform convergence on Y^X is finer than the topology of proximal convergence.*

Proof. Let $f \in Y^X$, and let A_i, B_i $(1 \leqslant i \leqslant m)$ be subsets of X, Y, respectively, such that $f \in \bigcap_{i=1}^{m} U_{A_i,B_i}$. Then for each i we have $f(A_i) \bowtie B_i$ in Y, so there exists $U_i \in \mathcal{U}$ such that $f(A_i) \times B_i \subset {\sim} U_i$. Let

$$U = \bigcap_{i=1}^{m} U_i,$$

which belongs to $\mathcal{U}$. Then

$$f(A_i) \times B_i \subset {\sim} U \quad (1 \leqslant i \leqslant m).$$

Pick $V, W \in \mathcal{U}$ such that (U, V, W) is a 3-chain, and consider any $g \in W_X(V)[f]$. Given $x \in A_i$ and $y \in B_i$, suppose that $(g(x), y) \in V$. Since $(f, g) \in W_X(V)$, we have $(f(x), g(x)) \in V$ and therefore $(f(x), y) \in V^2 \subset U$, a contradiction. Hence, in fact, $(g(x), y) \in {\sim} W$. It follows that $g(A_i) \times B_i \subset {\sim} W$, so $g(A_i) \bowtie B_i$ in Y. Hence $W_X(V)[f] \subset \bigcap_{i=1}^{m} U_{A_i,B_i}$. Thus every τ_p-neighbourhood of f in Y^X contains some $\tau_{\mathcal{W}}$-neighbourhood of f. $\square$

Corollary 2. *If X is an inhabited set and Y is a uniform space, then uniform convergence of a net in Y^X implies proximal convergence.*

Proof. Observing that uniform and proximal convergence are equivalent, respectively, to convergence in the topologies $\tau_{\mathcal{W}}$ and τ_p, apply Proposition 6. $\square$

Proposition 7. *Let X be an inhabited set, and $(Y, \mathcal{U})$ a totally bounded uniform space. Then the topologies τ_p and $\tau_{\mathcal{W}}$ coincide.*

Proof. In view of Proposition 6, it is enough to prove that τ_p is finer than $\tau_{\mathcal{W}}$. Given an entourage U of Y, construct a 5-chain $(U_1, U_2, U_3, U_4, U_5)$ of symmetric entourages of Y such that $U_2^3 \subset U_1 = U$ and $U_4^3 \subset U_3$. Choose $x_1, \ldots, x_m$ in X such that $Y = Y_1 \cup \cdots \cup Y_m$, where $Y_i = U_4[f(x_i)]$, and then set $X_i = f^{-1}(Y_i)$. For $1 \leqslant i, j \leqslant m$ construct c_{ij} such that

$$c_{ij} = 0 \Longrightarrow \left(f(x_i), f(x_j)\right) \in U_2,$$
$$c_{ij} = 1 \Longrightarrow \left(f(x_i), f(x_j)\right) \in {\sim} U_3.$$

For each (i, j) with $c_{ij} = 1$ we have $Y_i \bowtie Y_j$. To see this, consider such i, j and an element $\left(y, y'\right)$ of $Y_i \times Y_j$, and suppose that $\left(y, y'\right) \in U_4$. Then $(f(x_i), y) \in U_4$ and $\left(y', f(x_j)\right) \in U_4$; so $\left(f(x_i), f(x_j)\right) \in U_4^3 \subset U_3$, which is absurd since $c_{ij} = 1$. Hence $\left(y, y'\right) \in {\sim} U_5$. It follows that

$Y_i \times Y_j \subset \sim U_5$ and therefore that $Y_i \bowtie Y_j$. Hence

$$\bigcap_{\{(i,j):c_{ij}=1\}} U_{X_i,Y_j}$$

is a τ_p-neighbourhood of f. Consider any g in this neighbourhood and any $x \in X$. Choose i such that $f(x) \in Y_i = U_4[f(x_i)]$ and therefore $x \in X_i$. Choose also j such that $g(x) \in Y_j = U_4[f(x_j)]$. If $c_{ij} = 1$, then our choice of g ensures that $g(X_i) \bowtie Y_j$; so $g(x) \notin Y_j$, a contradiction. Thus $c_{ij} = 0$; whence $\big(f(x_i), f(x_j)\big) \in U_2$ and, by symmetry, $(f(x_j), f(x_i)) \in U_2$. Since $\big(g(x), f(x_j)\big) \in U_4 \subset U_2$, it follows that $(g(x), f(x_i)) \in U_2^2$; from which, as $(f(x_i), f(x)) \in U_4$, we obtain

$$(g(x), f(x)) \in U_2^3 \subset U_1 = U.$$

Since g and x are arbitrary, we conclude that

$$f \in \bigcap_{\{(i,j):c_{ij}=1\}} U_{X_i,Y_j} \subset W_X(U)[f].$$

It follows that every $\tau_{\mathcal{W}}$-open set is τ_p-open. □

Corollary 3. *Let X be an inhabited set, and $(Y, \mathcal{U})$ a totally bounded uniform space. Then proximal convergence of nets in Y^X is equivalent to uniform convergence.*

Proof. Let $(f_n)_{n \in D}$ be a net that converges proximally to f in Y^X. Then by Proposition 3, this net converges to f in the topology τ_p. By Proposition 7, the net therefore converges to f in the topology $\tau_{\mathcal{W}}$; in other words, it converges uniformly to f. □

Proposition 8. *Let X be an inhabited set, and $(Y, \mathcal{U})$ a uniform space. Then uniform convergence in Y^X is equivalent to $\bowtie_{\mathcal{W}}$-convergence.*

Proof. Let the net $(f_n)_{n \in D}$ converge uniformly to f in Y^X, let $S \subset Y^X$, and let $f \bowtie_{\mathcal{W}} S$. Then there exists $U \in \mathcal{U}$ such that $\{f\} \times S \subset \neg W_X(U)$. Choose a symmetric $V \in \mathcal{U}$ such that $V \circ V \subset U$. Then there exists $N \in D$ such that $(f, f_n) \in W_X(V)$ for all $n \succcurlyeq N$. Consider such n, and g in S. If $(f_n, g) \in W_X(V)$, then $(f, g) \in W_X(V) \circ W_X(V) \subset W_X(U)$, a contradiction of our choice of U. Hence $(f_n, g) \in \neg W_X(V)$. It follows that for all $n \succcurlyeq N$ we have $\{f_n\} \times S \subset \neg W_X(V)$ and therefore $f_n \bowtie_{\mathcal{W}} S$.

Conversely, suppose that the net $(f_n)_{n \in D}$ is $\bowtie_{\mathcal{W}}$-convergent to f in Y^X. Fix $U \in \mathcal{U}$ and choose $V \in \mathcal{U}$ such that $X \times X = U \cup \neg V$. Define

$$S = \{f_n : (f, f_n) \in \neg W_X(V)\}. \tag{3}$$

Then $\{f\} \times S \subset \neg W_X(V)$, so $f \bowtie_{\mathcal{W}} S$. Hence there exists n such that $f_n \bowtie_{\mathcal{W}} S$ for all $n \succcurlyeq N$. It follows that for all such n we have $(f, f_n) \notin \neg W_X(V)$. Now fix $x \in X$ and let $n \succcurlyeq N$. If $(f(x), f_n(x)) \in \neg V$, then $(f, f_n) \in \neg W_X(V)$, a contradiction. Hence $(f(x), f_n(x)) \notin \neg V$ and therefore $(f(x), f_n(x)) \in U$. Thus $(f, f_n) \in W_X(U)$ for all $n \succcurlyeq N$. □

A natural classical approach to proving the second half of Proposition 8 for sequences of functions goes as follows. Let $(f_n)_{n \geqslant 1}$ be a sequence that is $\bowtie_{\mathcal{W}}$-convergent to f in Y^X.

Fix $U \in \mathcal{U}$ and suppose that

$$\neg\exists_N \forall_{n \geqslant N} \left((f, f_n) \in W_X(U)\right).$$

Then we can find a strictly increasing sequence $(n_k)_{k \geqslant 1}$ such that $\left(f, f_{n_k}\right) \in \neg W_X(U)$ for all k. It follows that

$$\{f\} \times \left\{f_{n_k} : k \geqslant 1\right\} \subset \neg W_X(U)$$

and hence that

$$f \bowtie_{\mathcal{W}} \left\{f_{n_k} : k \geqslant 1\right\}.$$

Since $(f_n)_{n \geqslant 1}$ is $\bowtie_{\mathcal{W}}$-convergent to f, there exists N such that

$$f_n \bowtie_{\mathcal{W}} \left\{f_{n_k} : k \geqslant 1\right\} \quad (n \geqslant N).$$

Choosing j such that $n_j > N$, we obtain the contradiction $f_{n_j} \bowtie_{\mathcal{W}} f_{n_j}$. Thus

$$\exists_N \forall_{n \geqslant N} \ \left((f, f_n) \in W_X(U)\right).$$

Since $U \in \mathcal{U}$ is arbitrary, we conclude that $(f_n)_{n \geqslant 1}$ converges uniformly to f.

This indirect proof contrasts sharply with our constructive one, in which, given $U \in \mathcal{U}$ and considering the set S introduced at (3), we are able to produce the desired index N such that $(f, f_n) \in W_X(U)$ for all $n \succcurlyeq N$.

Proposition 9. *Let X be an inhabited set, and $(Y, \mathcal{U})$ a uniform space. Then $\bowtie_{Y^X} \subset \bowtie_{\mathcal{W}}$ in the sense that*

$$\forall_{f \in Y^X} \forall_{S \subset Y^X} \left(f \bowtie_{Y^X} S \Rightarrow f \bowtie_{\mathcal{W}} S\right).$$

Proof. Let $f \bowtie_{Y^X} S$. Then there exists a τ_p-open set $A \subset Y^X$ such that $f \in A \subset \neg S$. By Proposition 6, there exists $U \in \mathcal{U}$ such that $W_X(U)[f] \subset A$, so $W_X(U)[f] \subset \neg S$. It follows that if $g \in S$, then $(f, g) \notin W_X(U)$. Thus $\{f\} \times S \subset \neg W_X(U)$, and therefore $f \bowtie_{\mathcal{W}} S$. $\square$

What can we say about uniform spaces for which $\bowtie_{\mathcal{W}} \subset \bowtie_{Y^X}$ and hence, by the preceding proposition, the pre-apartnesses $\bowtie_{Y^X}$ and $\bowtie_{\mathcal{W}}$ coincide?

Proposition 10. *Let X be an inhabited set, and $(Y, \mathcal{U})$ a uniform space such that*

$$\forall_{f \in Y^X} \forall_{S \subset Y^X} \left(f \bowtie_{\mathcal{W}} S \Rightarrow f \bowtie_{Y^X} S\right).$$

Then $\bowtie_{Y^X}$-convergence, uniform convergence, and proximal convergence are equivalent.

Proof. We already know (from Corollary 2) that uniform convergence implies proximal convergence, and (from Corollary 1) that proximal convergence implies $\bowtie_{Y^X}$-convergence. Thus it suffices to prove that $\bowtie_{Y^X}$-convergence implies uniform convergence. Accordingly, let $(f_n)_{n \in D}$ be a net that is $\bowtie_{Y^X}$-convergent to f in Y^X. Let $S \subset Y^X$ and $f \bowtie_W S$. By hypothesis, $f \bowtie_{Y^X} S$; so there exists $N \in D$ such that $f_n \bowtie_{Y^X} S$ for all $n \succcurlyeq N$. Proposition 9 now shows that $f_n \bowtie_{\mathcal{W}} S$ for all $n \succcurlyeq N$. Hence $(f_n)_{n \in D}$ is $\bowtie_{\mathcal{W}}$-convergent to f, and therefore, by Proposition 8, uniformly convergent to f. $\square$

Nachman has shown classically that proximal convergence need not imply uniform convergence [13]. Consequently, when Y is a uniform space, we have no guarantee that $\bowtie_{\mathcal{W}} \subset \bowtie_{Y^X}$.

Corollary 4. *Under the hypotheses of Proposition* 10, *the space* (Y^X, τ_p) *is topologically consistent.*

Proof. This follows immediately from Proposition 10 and Corollary 1. □

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